Toric orbits as Lagrangian submanifolds

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The moment map:

2. We consider a $2n$-dimensional symplectic manifold $(M^{2n}, \omega)$ with a Hamiltonian action of the $n$-torus $T^n$. This is generated by a moment map

$$\Phi : M \to \mathbb{R}^n, \quad x \mapsto (H_1(x), \ldots, H_n(x)),$$

where the function $H_i : M \to \mathbb{R}$ generates the $i$th circle action via $\omega(\xi_i, \cdot) = dH_i$. (The vector field $\xi_i$ is tangent to the orbits of the $i$th action.)

![Diagram of the moment map](image)

**Figure**: The simplest compact example is $S^2$ with $S^1$ acting by rotation about a vertical axis. $\Phi$ is the height function: $(x_1, x_2, x_3) \mapsto x_3$. The $S^1$-orbits are horizontal circles, the fibers $\Phi^{-1}(t)$ of the moment map.
The moment image:

\[ (z_1, z_2) \rightarrow (|z_1^2|, |z_2^2|) \]

Figure: \( S^2 \times S^2 \) with the product action of \( S^1 \times S^1 \): the moment image \( \Phi(S^2 \times S^2) \) is the square \([0, 1] \times [0, 1] \). \( \Phi^{-1}(x) = \) circle orbit.

Figure: In polar coordinates \((r, \theta)\) on \( \mathbb{R}^2 \) with action \( t \cdot (r, \theta) = (r, \theta + t) \) the moment map is \((r, \theta) \mapsto r^2\): hence \( z \mapsto |z|^2 =: t \) for \( z = re^{2\pi it} \in \mathbb{C} \). Thus the open ball \( B^4 \subset \mathbb{C}^2 \equiv \mathbb{R}^4 \) has moment image equal to a triangle.
The moment image $\Phi(M) = \Delta_M$ is a simple convex polytope. By Delzant's theorem, $\Delta_M$ determines the toric manifold $(M, T^n, \omega)$.

The shape of $\Delta_M$ depends on the chosen identification of $T^n$ with the product $S^1 \times \cdots \times S^1$, together with the additive constants chosen for the Hamiltonians $H_i$. Thus $\Delta_M \subset \mathbb{R}^n$ is well defined modulo the action of the integral affine group $\text{Aff} (n, \mathbb{Z}) \cong \text{SL} (n, \mathbb{Z}) \times \mathbb{R}^n$ (symmetries of the torus). The relevant geometry is integral affine geometry.

Figure: These triangles are all affine equivalent, describing the same toric manifold, namely the complex projective plane $\mathbb{C}P^2$. 
Toric fibers:

The moment map (for closed $M$) is in fact the quotient map

$$\Phi : M \to M/T^n \equiv \Delta_M.$$  

Each interior fiber $L_u := \Phi^{-1}(u), u \in \text{Int}\Delta_M$ is a Lagrangian torus, i.e. $\omega|_{L_u} \equiv 0$. No two are equivalent under the action of the Hamiltonian group $\text{Ham}(M, \omega)$ (the identity component of the symplectomorphism group) because they bound discs of different areas:

![Diagram of a triangle with vertices labeled P, u, and Q, and dashed lines connecting them.](image)

**Figure:** $\Phi^{-1}(P) \cong D^2(a) \times S^1$ and $\Phi^{-1}(Q) \cong D^2(b) \times S^1$, where $D^2(a), D^2(b)$ are discs whose areas $a, b$ equal the affine lengths of the lines $P, Q$. (Note: $\omega$ restricts on $D^2 \times S^1$ to the pullback of the area form of the disc.) Also $L_u = S^1(a) \times S^1(b)$
Displaceability:

Which toric fibers are displaceable by a Hamiltonian isotopy? i.e. when is there $\phi \in \text{Ham}(M)$ such that $\phi(L_u) \cap L_u = \emptyset$?

(Entov–Polterovich [EP09]): There is always at least one such fiber.

We can detect non-displaceable fibers using (variants) of Floer homology. In general, this is hard to define and needs auxiliary structures (certain 1-forms $b$). There has been much work on this by Cho, Cho–Oh, Fukaya–Oh–Ohta–Ono [FOOO10], Woodward, Wilson–Woodward [WW], Abreu–Macarini [AM13]...

**FACT:** If the Floer homology $HF(L_u, b)$ is defined and nonzero for some $b$, then $L_u$ is non-displaceable.

**FACT:** $HF(L_u, b)$ can be nonzero only if $u$ has at least two (and usually three) closest facets, where “closeness” is measured using affine distance $d_{aff}$. (More precisely, $u$ must be a critical point of the Landau–Ginzburg potential.)
Examples:

The facet $F$ with outward conormal $\eta$ has equation.

$$F : \eta \cdot u = \eta_1 x_1 + \eta_2 x_2 = \kappa$$

The affine distance $d_{aff}(u, F)$ between the point $u = (u_1, u_2)$ and $F$ is $d_{aff}(u, F) = \kappa - (\eta_1 u_1 + \eta_2 u_2)$.

Figure: On the left is $\mathbb{C}P^2$ with a (unique) non-displaceable fiber at the center of gravity $u_0$ of $\Delta$. Note that $d_{aff}(u_0, F_1) = 1 = d_{aff}(u_0, F_3)$. The other two figures are one point blow-ups, a small one in the middle with two non-displaceable points, and a large blow-up on the right with one. These four heavy dots mark the only points in these examples where the Floer homology does not vanish.
Displacing fibers

A probe $P$ is a line segment in $\Delta \subset \mathbb{R}^n$

- starting at a point $q$ in the interior of a facet $F$ and
- whose direction vector $\nu$ can be completed to an integral basis for $\mathbb{R}^n$ by adding vectors parallel to $F$.

**FACT:** A point $u \in \text{Int} \Delta$ is displaceable (i.e. the corresponding toric fiber $L_u$ can be moved off itself by a Hamiltonian isotopy) if it lies less than halfway along a probe: cf. McDuff [Mc11i].

\[ \Phi^{-1}(P) = D^2(L) \times S^1 \subset M, \text{ with symplectic form pulled back from the disc, while } \Phi^{-1}(u) = D^2(a) \times S^1 \text{ if } u \text{ is a distance } a \text{ along the probe. So if } a < L/2 \text{ we can displace } S^1(a) \text{ to } \phi(S^1(a)) \text{ inside } D^2(L) \text{ by an area-preserving map, and then extend this deformation to } M. \]
Some Probes and non Probes:

Figure: probes are solid lines with arrows, non probes are dotted: either the direction or starting point is wrong.

In particular, the downward diagonal line has direction $(1, -1)$, and it starts on a facet with direction $(1, 1)$: but these two vectors have determinant $= 2$ so do not form an integral basis.

All the points inside this figure except for those on the very heavy horizontal line are displaceable by probes. Those on the line are not: in fact they all have nontrivial Floer homology [FOOO12].

Gonzalez–Woodward [GW12] interpret the non displaceable points detected by probes in terms of the minimal model program.
Floer homology and Probes:

Figure: Floer homology is calculated by an iterative construction: since the points on the horizontal are equidistant from two nearest parallel lines we can look at the next closest facets, adding a ghost facet to make each such point \( u \) equidistant from three facets (here \( F_1, F_2, F_3 \)). Abreu–Macarini explain this in terms of symplectic reduction.

Figure: A point whose Floer homology vanishes, but is inaccessible by probes: the potential probes that reach this point either start at a vertex or (as in the case of the vertical line) have a bad direction. This point can be displaced by an extended probe; cf. Abreu–Borman–McDuff.
Some Open Problems:

**Problem 1:** A monotone toric manifold, scaled so that $[\omega] = c_1$, has a unique interior integral point. It is the unique point with nontrivial Floer homology. Is this the only non displaceable point? There is a known $n = 6$-dimensional example where this point cannot be displaced by probes. cf. McDuff [Mc11ii]

**Problem 2:** Probes (and Floer invariants) can also be used to investigate similar questions in toric orbifolds. These tend to have fewer displaceable fibers and a richer set of invariants.

Figure: On the left the $A_3$-singularity: points on the dark rays have nontrivial Floer invariants, and the rest are displaceable by (extended) probes. On the right the resolved singularity: here all Floer invariants vanish, but no ways are known to displace the points on the dotted lines.
A few References:


